

Classification of families of pr- and epr-sequences

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Abstract

This paper establishes new restrictions for attainable enhanced principal rank characteristic sequences (epr-sequences). These results are then used to classify two related families of sequences that are attainable by a real symmetric matrix: the family of principal rank characteristic sequences (pr-sequences) not containing three consecutive 1s and the family of epr-sequences which contain an N in every subsequence of length 3.

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1 Introduction

Given an $n \times n$ symmetric matrix B over a field F , the *principal rank characteristic sequence* (abbreviated pr-sequence) of B is defined as $\text{pr}(B) = r_0 r_1 \cdots r_n$, where

$$r_k = \begin{cases} 1 & \text{if } B \text{ has a nonzero principal minor of order } k, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

while $r_0 = 1$ if and only if B has a 0 diagonal entry [2]; the *order* of a minor is k if it is the determinant of a $k \times k$ submatrix.

The *principal minor assignment problem*, introduced in [5], asks the following question: can we find an $n \times n$ matrix with prescribed principal minors? As a simplification of the principal minor assignment problem, Brualdi et al. [2] introduced the pr-sequence of a real symmetric matrix as defined above. An attractive result obtained in [2] is the requirement

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that a pr-sequence that can be realized by a real symmetric matrix cannot contain the subsequence 001, meaning that in the pr-sequence of such matrix, the presence of the subsequence 00 forces 0s from that point forward. This result was later generalized by Barrett et al. [1] for symmetric matrices over any field; this led them to the study of symmetric matrices over various fields, where, among other results, a characterization of the pr-sequences that can be realized by a symmetric matrix over a field of characteristic 2 was obtained. Although not deeply studied, the family of pr-sequences not containing three consecutive 1s were of interest in [2], since the pr-sequences of the principal submatrices of a matrix realizing a pr-sequence not containing three consecutive 1s possess the rare property of being able to inherit the majority of the 1s of the original sequence; this family will be one of the central themes of this paper.

Due to the limitations of the pr-sequence, which only records the presence or absence of a full-rank principal submatrix of each possible order, Butler et al. [3] introduced the *enhanced principal rank characteristic sequence* (abbreviated epr-sequence) of an $n \times n$ symmetric matrix B over a field F , denoted by $\text{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$, where

$$\ell_k = \begin{cases} \text{A} & \text{if all the principal minors of order } k \text{ are nonzero;} \\ \text{S} & \text{if some but not all the principal minors of order } k \text{ are nonzero;} \\ \text{N} & \text{if none of the principal minors of order } k \text{ are nonzero, i.e., all are zero.} \end{cases}$$

A (pr- or epr-) sequence is said to be *attainable* over a field F provided that there exists a symmetric matrix $B \in F^{n \times n}$ that attains it; otherwise, we say that it is *unattainable*. Among other results, techniques to construct attainable epr-sequences were presented in [3], as well as necessary conditions for an epr-sequence to be attainable by a symmetric matrix, with many of them asserting that subsequences such as **NSA**, **NAN** and **NAS**, among others, cannot occur in epr-sequences over certain fields. Continuing the study of epr-sequences, Fallat et al. [4] characterized all the epr-sequences that are attainable by skew-symmetric matrices.

In this paper the study of pr- and epr-sequences of symmetric matrices is continued. Section 2 establishes new restrictions for epr-sequences to be attainable over certain fields. The results from Section 2 are then implemented in Section 3, where, for real symmetric matrices, we classify all the attainable pr-sequences not containing three consecutive 1s. Using this classification, in Section 4, a related family of attainable epr-sequences is classified, namely those that contain an **N** in every subsequence of length 3. We then conclude with Proposition 4.6, where we highlight an interesting property exhibited by the vast majority of attainable pr-sequences not containing three consecutive 1s; that is, the property of being associated with a unique attainable epr-sequence.

A pr-sequence and an epr-sequence are *associated* with each other if a matrix (which may not exist) attaining the epr-sequence also attains the pr-sequence. A subsequence that does not appear in an attainable sequence is *forbidden* (and we may also say that it is *prohibited*). Moreover, a sequence is said to have *order* n if it corresponds to a matrix of order n , while a subsequence has *length* n if it consists of n terms.

Let $B = [b_{ij}]$ and let $\alpha, \beta \subseteq \{1, 2, \dots, n\}$. Then the submatrix lying in rows indexed by α , and columns indexed by β , is denoted by $B[\alpha, \beta]$; if $\alpha = \beta$, then $B[\alpha, \alpha]$ is abbreviated to $B[\alpha]$. The matrices 0_n , I_n and J_n are the matrices of order n denoting the zero matrix, the identity matrix and the all-1s matrix, respectively. The direct sum of two matrices B and

C is denoted by $B \oplus C$. Given a graph G , $A(G)$ denotes the adjacency matrix of G , while P_n and C_n denote the path and cycle, respectively, on n vertices.

1.1 Results cited

The purpose of this section is to list results we will cite frequently, and assign abbreviated nomenclature to some of them.

Theorem 1.1. [2, Theorem 2.7] Suppose B is a nonsingular real symmetric matrix with $\text{pr}(B) = r_0]r_1 \cdots r_n$. Let $\text{pr}(B^{-1}) = r'_0]r'_1 \cdots r'_n$. Then $r'_n = r_n = 1$, while for each i with $1 \leq i \leq n-1$, $r'_i = r_{n-i}$. Finally, $r'_0 = 1$ if and only if B has some principal minor of order $n-1$ that is zero.

Theorem 1.2. [2, Theorem 4.4] (00 Theorem) *Let B be a real symmetric matrix. Let $\text{pr}(B) = r_0]r_1 \cdots r_n$ and suppose that, for some k with $0 \leq k \leq n-2$, $r_{k+1} = r_{k+2} = 0$. Then $r_i = 0$ for all $i \geq k+1$. In particular, $r_n = 0$, so that B is singular.*

Theorem 1.3. [2, Theorem 6.5] (0110 Theorem) *Suppose $n \geq 4$ and $\text{pr}(B) = r_0]r_1 \cdots r_n$. If, for some k with $1 \leq k \leq n-3$, $r_k = r_{k+3} = 0$, then $r_i = 0$ for all $k+3 \leq i \leq n$. In particular, B is singular.*

A generalization of Theorem 1.2 in [1] led to an analogous result for epr-sequences over any field:

Theorem 1.4. [3, Theorem 2.3] (NN Theorem) *Suppose B is a symmetric matrix over a field F , $\text{epr}(B) = \ell_1\ell_2 \cdots \ell_n$, and $\ell_k = \ell_{k+1} = \mathbf{N}$ for some k . Then $\ell_i = \mathbf{N}$ for all $i \geq k$. (That is, if an epr-sequence of a matrix ever has NN, then it must have Ns from that point forward.)*

Theorem 1.5. [3, Theorem 2.4] (Inverse Theorem) *Suppose B is a nonsingular symmetric matrix over a field F . If $\text{epr}(B) = \ell_1\ell_2 \cdots \ell_{n-1}\mathbf{A}$, then $\text{epr}(B^{-1}) = \ell_{n-1}\ell_{n-2} \cdots \ell_1\mathbf{A}$.*

Each instance of \cdots below is permitted to be empty.

Proposition 1.6. [3, Proposition 2.5] *The epr-sequence $\mathbf{SN} \cdots \mathbf{A} \cdots$ is forbidden for symmetric matrices over any field.*

We say that $\mathbf{SN} \cdots \mathbf{A} \cdots$ is prohibited when referencing Proposition 1.6.

Theorem 1.7. [3, Theorem 2.6] (Inheritance Theorem) *Suppose that B is a symmetric matrix over a field F , $m \leq n$, and $1 \leq i \leq m$.*

1. *If $[\text{epr}(B)]_i = \mathbf{N}$, then $[\text{epr}(C)]_i = \mathbf{N}$ for all $m \times m$ principal submatrices C .*
2. *If $[\text{epr}(B)]_i = \mathbf{A}$, then $[\text{epr}(C)]_i = \mathbf{A}$ for all $m \times m$ principal submatrices C .*
3. *If $[\text{epr}(B)]_m = \mathbf{S}$, then there exist $m \times m$ principal submatrices C_A and C_N of B such that $[\text{epr}(C_A)]_m = \mathbf{A}$ and $[\text{epr}(C_N)]_m = \mathbf{N}$.*
4. *If $i < m$ and $[\text{epr}(B)]_i = \mathbf{S}$, then there exists an $m \times m$ principal submatrix C_S such that $[\text{epr}(C_S)]_i = \mathbf{S}$.*

Corollary 1.8. [3, Corollary 2.7] *No symmetric matrix over any field can have NSA in its epr-sequence. Further, no symmetric matrix over any field can have the epr-sequence $\cdots \text{ASN} \cdots \mathbf{A} \cdots$.*

Corollary 1.8 will be invoked by just stating that NSA or $\cdots \text{ASN} \cdots \mathbf{A} \cdots$ is prohibited.

If B is a matrix with a nonsingular principal submatrix $B[\alpha]$, $B/B[\alpha]$ denotes the Schur complement of $B[\alpha]$ in B .

Theorem 1.9. [3, Proposition 2.13] (Schur Complement Theorem) *Suppose B is a symmetric matrix over a field of characteristic not 2 with $\text{rank } B = m$. Let $B[\alpha]$ be a nonsingular principal submatrix of B with $|\alpha| = k \leq m$, and let $C = B/B[\alpha]$. Then the following results hold.*

1. C is an $(n - k) \times (n - k)$ symmetric matrix.
2. Assuming the indexing of C is inherited from B , any principal minor of C is given by

$$\det C[\gamma] = \det B[\gamma \cup \alpha] / \det B[\alpha].$$

3. $\text{rank } C = m - k$.
4. Any nonsingular principal submatrix of B of order at most m is contained in a nonsingular principal submatrix of order m .

Theorem 1.10. [3, Theorem 2.14] *Neither the epr-sequences NAN nor NAS can occur as a subsequence of the epr-sequence of a symmetric matrix over a field of characteristic not 2.*

We will refer to Theorem 1.10 by simply stating that NAN or NAS is prohibited, while Theorem 1.11 below is referenced by stating that ANS ‘must be initial.’

Theorem 1.11. [3, Theorem 2.15] *In the epr-sequence of a symmetric matrix over a field of characteristic not 2, the subsequence ANS can only occur as the initial subsequence.*

2 Restrictions on attainable epr-sequences

In this section, we establish new restrictions on attainable epr-sequences. We begin with restrictions that apply to fields of characteristic not 2. For convenience, given a matrix B , we adopt some of the notation in [2], and denote with $B_{i_1 i_2 \dots i_k}$, the principal minor $\det(B[\{i_1, i_2, \dots, i_k\}])$.

Proposition 2.1. *Let $n \geq 6$. Then no $n \times n$ symmetric matrix over a field of characteristic not 2 has an epr-sequence starting NSNA \cdots .*

Proof. Let $B = [b_{ij}]$ be an $n \times n$ symmetric matrix over a field of characteristic not 2 and let $\text{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$. Suppose to the contrary that $\text{epr}(B) = \text{NSNA} \cdots$. Since $\ell_3 = \mathbf{N}$, and because $B_{pqr} = 2b_{pq}b_{pr}b_{qr}$ for any distinct $p, q, r \in \{1, 2, \dots, n\}$, $B[\{1, 2, 3\}]$ and $B[\{4, 5, 6\}]$ must each contain a zero off-diagonal entry. Moreover, since $\ell_4 = \mathbf{A}$, 0_3 is not a principal submatrix of B , implying that $B[\{1, 2, 3\}]$ and $B[\{4, 5, 6\}]$ must each contain a nonzero

off-diagonal entry. Since $\{1, 2, 3\}$ and $\{4, 5, 6\}$ are disjoint, and because a simultaneous permutation of the rows and columns of a matrix has no effect on its determinant, we may assume, without loss of generality, that $b_{12} = b_{56} = 0$ and that b_{13}, b_{46} are nonzero. Similarly, since $\{1, 2, 3\}$ and $\{4, 5, 6\}$ are disjoint, and because multiplication of any row and column of a matrix by a nonzero constant preserves the rank of every submatrix, we may also assume, without loss of generality, that $b_{13} = b_{46} = 1$. We consider two cases.

Case 1: $b_{14} = 0$. Since $\ell_4 = \mathbf{A}$, $(b_{15}b_{24})^2 = B_{1245} \neq 0$; it follows that b_{15} and b_{24} are nonzero. Since $\ell_3 = \mathbf{N}$, $B_{135} = 2b_{15}b_{35} = 0$; hence, $b_{35} = 0$. Since $B[\{3, 5, 6\}] \neq 0_3$, $b_{36} \neq 0$. Since $2b_{16}b_{36} = B_{136} = 0$, $b_{16} = 0$. Then, as $B[\{1, 2, 6\}] \neq 0_3$, $b_{26} \neq 0$. It follows that $B_{246} = 2b_{24}b_{26} \neq 0$, a contradiction to $\ell_3 = \mathbf{N}$, implying that it is impossible to have $b_{14} = 0$.

Case 2: $b_{14} \neq 0$. Since $2b_{14}b_{34} = B_{134} = 0$, and because $2b_{14}b_{16} = B_{146} = 0$, $b_{34} = b_{16} = 0$. Since $B[\{1, 2, 6\}] \neq 0_3$, $b_{26} \neq 0$. Since $2b_{24}b_{26} = B_{246} = 0$, $b_{24} = 0$. Since $(b_{14}b_{23})^2 = B_{1234} \neq 0$, $b_{23} \neq 0$. Then, as $2b_{23}b_{26}b_{36} = B_{236} = 0$, $b_{36} = 0$. It follows that $B_{1356} = 0$, a contradiction to $\ell_4 = \mathbf{A}$. \square

It should be noted that **NSNA** and **NSNAA** are attainable by $A(P_4)$ and $A(C_5)$, respectively [3], but this does not contradict Proposition 2.1, which requires $n \geq 6$.

Proposition 2.2. *Let B be a symmetric matrix over a field of characteristic not 2 and $\text{epr}(B) = \ell_1\ell_2 \cdots \ell_n$. Then **NSNA** cannot occur as a subsequence of $\ell_1\ell_2 \cdots \ell_{n-2}$.*

Proof. If $n \leq 5$, the result follows vacuously. So, assume $n \geq 6$. Suppose to the contrary that **NSNA** occurs as a subsequence of $\ell_1\ell_2 \cdots \ell_{n-2}$ and that $\ell_k\ell_{k+1}\ell_{k+2}\ell_{k+3} = \mathbf{NSNA}$, for some k with $1 \leq k \leq n-5$. By Proposition 2.1, $k \geq 2$, and, by the NN Theorem, $\ell_{k-1} \neq \mathbf{N}$; it follows that B has a $(k-1) \times (k-1)$ nonsingular principal submatrix, say $B[\alpha]$. By the Schur Complement Theorem, $B/B[\alpha]$ has an epr-sequence starting $\mathbf{NXNAYZ} \cdots$, where $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \{\mathbf{A}, \mathbf{S}, \mathbf{N}\}$. The NN Theorem and the fact that **NAN** is prohibited imply that $\mathbf{X} = \mathbf{S}$; hence, $\text{epr}(B)$ starts **NSNAYZ** \cdots , a contradiction to Proposition 2.1. \square

With the next result, we generalize (and provide a simpler proof of) [3, Proposition 2.11].

Proposition 2.3. *Suppose B is a symmetric matrix over a field of characteristic not 2, $\text{epr}(B) = \ell_1\ell_2 \cdots \ell_n$ and $\ell_k\ell_{k+1}\ell_{k+2} = \mathbf{SAN}$ for some k . Then $\ell_j = \mathbf{N}$ for all $j \geq k+2$.*

Proof. If $n = 3$, we are done. Suppose $n > 3$. Suppose that $\ell_k\ell_{k+1}\ell_{k+2} = \mathbf{SAN}$ for some k with $1 \leq k \leq n-2$. If $k = n-2$, we are done. Suppose $k < n-2$. By [3, Corollary 2.10], which prohibits **SANA**, $\ell_{k+3} \neq \mathbf{A}$. Since **ANS** must be initial, $\ell_{k+3} \neq \mathbf{S}$. Hence, $\ell_{k+3} = \mathbf{N}$. The desired conclusion now follows from the NN Theorem. \square

We now confine our attention to real symmetric matrices. The next result is immediate from Theorem 1.3.

Proposition 2.4. *Let B be a real symmetric matrix and $\text{epr}(B) = \ell_1\ell_2 \cdots \ell_n$. Suppose $\ell_k = \ell_{k+3} = \mathbf{N}$ for some $k \geq 1$. Then $\ell_i = \mathbf{N}$ for all $i \geq k+3$. In particular, B is singular.*

We emphasize that Proposition 2.4 asserts that a sequence of the form $\cdots \mathbf{NXYN} \cdots \mathbf{Z} \cdots$, with $\mathbf{X}, \mathbf{Y} \in \{\mathbf{A}, \mathbf{S}, \mathbf{N}\}$ and $\mathbf{Z} \in \{\mathbf{A}, \mathbf{S}\}$, is not attainable by a real symmetric matrix.

Given a sequence $t_{i_1}t_{i_2} \cdots t_{i_k}, \overline{t_{i_1}t_{i_2} \cdots t_{i_k}}$ indicates that the sequence may be repeated as many times as desired (or it may be omitted entirely). According to [3, Proposition 2.17],

the sequence $\text{ANA}\bar{\text{A}}$ is attainable by a symmetric matrix over a field of characteristic 0. [3, Table 1] raises the following question: does a real symmetric matrix, with an epr-sequence starting $\text{ANA}\cdots$, always have epr-sequence $\text{ANA}\bar{\text{A}}$? The answer is affirmative; what follows makes this precise.

Proposition 2.5. *Any $n \times n$ real symmetric matrix with an epr-sequence starting $\text{ANA}\cdots$ is conjugate by a nonsingular diagonal matrix to one of $\pm(J_n - 2I_n)$. Furthermore, its epr-sequence is $\text{ANA}\bar{\text{A}}$.*

Proof. Let $B = [b_{ij}]$ be an $n \times n$ real symmetric matrix with an epr-sequence starting $\text{ANA}\cdots$. Notice that all the diagonal entries of B must have the same sign, as otherwise there would be a principal minor of order 2 that is nonzero. Let $C = [c_{ij}]$ be the matrix among B and $-B$ with all diagonal entries negative. Let $D = [d_{ij}]$ be the $n \times n$ diagonal matrix with $d_{11} = 1/\sqrt{-c_{11}}$ and $d_{jj} = \text{sign}(c_{1j})/\sqrt{-c_{jj}}$ for $j \geq 2$. Now, notice that every entry of DCD is ± 1 , every diagonal entry is -1 and every off-diagonal entry in the first row and the first column is 1. We now show that $DCD = J_n - 2I_n$. Since multiplication of any row and column of a matrix by a nonzero constant preserves the rank of every submatrix, $\text{epr}(DCD) = \text{epr}(C) = \text{epr}(B)$. Let $i, j \in \{2, 3, \dots, n\}$ be distinct, $\alpha = \{1, i, j\}$ and let a be the (i, j) -entry of DCD . A simple computation shows that $\det((DCD)[\alpha]) = (a + 1)^2$. Since every principal minor of order 3 of DCD is nonzero, $a = 1$. Then, as i and j were arbitrary, $DCD = J_n - 2I_n$. Then, as $C = B$ or $C = -B$, it follows that B is conjugate by a nonsingular diagonal matrix to one of $\pm(J_n - 2I_n)$, and that $\text{epr}(B) = \text{epr}(J_n - 2I_n) = \text{ANA}\bar{\text{A}}$ (see [3, Proposition 2.17]). \square

We are now in position to prove the following result.

Theorem 2.6. *Any epr-sequence of a real symmetric matrix containing ANA as a non-terminal subsequence is of the form $\bar{\text{A}}\text{ANAA}\bar{\text{A}}$.*

Proof. Let B be a real symmetric matrix containing ANA as a non-terminal subsequence. Let $\text{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose $\ell_{k+1}\ell_{k+2}\ell_{k+3} = \text{ANA}$ for some k with $0 \leq k \leq n - 4$. Since NAN and NAS are prohibited, $\ell_{k+4} = \text{A}$. If $k = 0$, the conclusion follows from Proposition 2.5; so, assume $k > 0$. Suppose $\ell_i \neq \text{A}$ for some i with $i < k + 1$. By the Inheritance Theorem, B has a (nonsingular) $(k + 4) \times (k + 4)$ principal submatrix B' whose epr-sequence $\ell'_1\ell'_2\cdots\ell'_{k+4}$ ends with ANAA and has $\ell'_i \neq \text{A}$. Then, by the Inverse Theorem, $\text{epr}((B')^{-1})$ starts with ANA and $\text{epr}((B')^{-1}) \neq \text{ANA}\bar{\text{A}}$, a contradiction to Proposition 2.5. Thus, $\text{epr}(B) = \bar{\text{A}}\text{ANAA}\ell_{k+5}\cdots\ell_n$, where $\ell_{k+5}\cdots\ell_n$ may not exist.

We now show that $\ell_{k+5}\cdots\ell_n = \bar{\text{A}}$. If $n = k + 4$, we are done; so, suppose $n > k + 4$. We proceed by contradiction, and consider two cases.

Case 1: $\ell_j = \text{N}$ for some $j > k + 4$. Since $\ell_k = \text{A}$, there exists a $k \times k$ principal submatrix of B , say $B[\alpha]$, that is nonsingular. Let $C = B/B[\alpha]$. By the Schur Complement Theorem, C has order $n - k$, $\text{epr}(C)$ starts $\text{ANA}\cdots$ and $\text{epr}(C)$ has an N in the $(j - k)$ -th position; hence, $\text{epr}(C) \neq \text{ANA}\bar{\text{A}}$, a contradiction to Proposition 2.5. It follows that a sequence containing ANA as a non-terminal subsequence cannot contain an N from that point forward, implying that any real symmetric matrix with an epr-sequence containing ANA is nonsingular.

Case 2: $\ell_j = \text{S}$ for some $j > k + 4$. By the Inheritance Theorem, B has a singular $j \times j$ principal submatrix whose epr-sequence contains ANA , which contradicts the assertion above.

We conclude that we must have $\ell_{k+5} \cdots \ell_n = \bar{A}$, which completes the proof. \square

It is natural to now ask, does Theorem 2.6 hold if **ANA** occurs at the end of the sequence? According to [3, Table 5], **SAANA** is attainable, answering the question negatively.

Theorem 2.7. *Let B be a real symmetric matrix with $\text{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$. Then **SNA** cannot occur as a subsequence of $\ell_1 \ell_2 \cdots \ell_{n-2}$.*

Proof. If $n \leq 4$, the result follows vacuously. So, assume $n > 4$. Suppose to the contrary that **SNA** occurs as a subsequence of $\ell_1 \ell_2 \cdots \ell_{n-2}$, and that $\ell_{k+1} \ell_{k+2} \ell_{k+3} = \text{SNA}$ for some k with $0 \leq k \leq n-5$. Since $\text{SN} \cdots \text{A} \cdots$ is prohibited, $k \geq 1$. Since **NAN** and **NAS** are prohibited, $\ell_{k+4} = \text{A}$. Then, as **ASNA** is prohibited, $\ell_k \neq \text{A}$. And, by Proposition 2.2, $\ell_k \neq \text{N}$; it follows that $\ell_k = \text{S}$. Thus, we have $\ell_k \cdots \ell_{k+4} = \text{SSNAA}$. We examine the three possibilities for ℓ_{k+5} .

Case 1: $\ell_{k+5} = \text{A}$. Now we have $\ell_k \cdots \ell_{k+5} = \text{SSNAAA}$. By the Inheritance Theorem, B has a $(k+5) \times (k+5)$ principal submatrix B' whose epr-sequence ends with **SXNAAA**, where $X \in \{\text{A}, \text{S}, \text{N}\}$. By the NN Theorem, $X \neq \text{N}$; and, by Proposition 2.3, $X \neq \text{A}$; it follows that $X = \text{S}$. By the Inverse Theorem, $\text{epr}((B')^{-1})$ contains **ANS** as a non-initial subsequence, a contradiction, since **ANS** must be initial. We conclude that $\ell_{k+5} \neq \text{A}$.

Case 2: $\ell_{k+5} = \text{N}$. Now we have $\ell_k \cdots \ell_{k+5} = \text{SSNAAN}$. Since $\ell_k = \text{S}$, B has a $k \times k$ nonsingular principal submatrix, say $B[\alpha]$. By the Schur Complement Theorem, $B/B[\alpha]$ has an epr-sequence starting **YNAAN**..., where $Y \in \{\text{A}, \text{S}, \text{N}\}$. By Theorem 2.6, $Y \neq \text{A}$; since $\text{SN} \cdots \text{A} \cdots$ is prohibited, $Y \neq \text{S}$; and, by the NN Theorem, $Y \neq \text{N}$. It follows that we must have $\ell_{k+5} \neq \text{N}$.

From Cases 1 and 2 we can deduce that the subsequence **SSNAAZ**, where $Z \in \{\text{A}, \text{N}\}$, cannot occur in the epr-sequence of a real symmetric matrix.

Case 3: $\ell_{k+5} = \text{S}$. Now we have $\ell_k \cdots \ell_{k+5} = \text{SSNAAS}$. By the Inheritance Theorem, B has a $(k+5) \times (k+5)$ principal submatrix with an epr-sequence ending with **SXNAAY**, where $X \in \{\text{A}, \text{S}, \text{N}\}$ and $Y \in \{\text{A}, \text{N}\}$. By the NN Theorem, $X \neq \text{N}$; and, by Proposition 2.3, $X \neq \text{A}$. It follows that $X = \text{S}$, which contradicts the assertion above. \square

As **NAN** is prohibited, we have the following corollary to Theorem 2.7.

Corollary 2.8. *The only way **SNA** can occur in the epr-sequence of a real symmetric matrix is in one of the two terminal sequences **SNA** or **SNAA**.*

We note that the epr-sequences **ANSSSNA** and **ANSSSNAA** are attainable [3, Table 1], implying that **SNA** is not completely prohibited in the epr-sequence of a real symmetric matrix. Theorem 2.6 and Corollary 2.8 lead to the following observation.

Observation 2.9. *Any epr-sequence of a real symmetric matrix that contains **NA** as a non-initial subsequence is of the form $\cdots \text{NA}\bar{A}$.*

The following results in this section will be of particular relevance to the main results in Sections 3 and 4.

Lemma 2.10. *Let n be even and B be a nonsingular $n \times n$ real symmetric matrix. Then $J_{\frac{n}{2}+1}$ is not a principal submatrix of B .*

Proof. For the sake of contradiction, suppose, without loss of generality, that $B[\{1, \dots, \frac{n}{2} + 1\}] = J_{\frac{n}{2}+1}$. Then the rank of the matrix consisting of the first $\frac{n}{2} + 1$ columns of B has rank less than $\frac{n}{2} + 1$; hence, B is singular, a contradiction. \square

Lemma 2.11. *Let $n \geq 8$ be even. Let B be an $n \times n$ nonsingular real symmetric matrix with every entry ± 1 and all entries in the first row, the first column, and the diagonal equal to 1. Suppose that $\text{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$ and that $\ell_4 = \mathbf{N}$. Then every row and column of B has at most $\frac{n}{2} - 1$ negative entries. Equivalently, every row and column of B has at least $\frac{n}{2} + 1$ positive entries.*

Proof. Suppose $B = [b_{ij}]$ contains a row with $\frac{n}{2}$ negative entries. Let $U = \{3, 4, \dots, \frac{n}{2} + 2\}$. Without loss of generality, suppose $b_{2j} = -1$ for all $j \in U$. We claim that $B[\{1\} \cup U] = J_{\frac{n}{2}+1}$. Suppose to the contrary that $B[\{1\} \cup U]$ contains a negative entry; without loss of generality, we may assume that this entry is b_{34} . It follows that $B[\{1, 2, 3, 4\}]$ is nonsingular, a contradiction to $\ell_4 = \mathbf{N}$; hence, $B[\{1\} \cup U] = J_{\frac{n}{2}+1}$. By Lemma 2.10, B is singular, a contradiction to the nonsingularity of B . We conclude that every row and column of B has at most $\frac{n}{2} - 1$ negative entries. \square

Theorem 2.12. *Let $n \geq 8$ be even and B be an $n \times n$ real symmetric matrix. Suppose that $\text{epr}(B) = \mathbf{A} \mathbf{N} \mathbf{S} \mathbf{N} \mathbf{S} \mathbf{N} \cdots$. Then B is singular.*

Proof. Suppose to the contrary that B is nonsingular. Let $B = [b_{ij}]$. By [2, Proposition 8.1], we may assume that every entry of B is ± 1 and all entries in the first row, the first column, and the diagonal are equal to 1. By Lemma 2.11, every row and column of B has at least $\frac{n}{2} + 1$ positive entries. Because a simultaneous permutation of the rows and columns of a matrix has no effect on its determinant, we may assume, without loss of generality, that the first $\frac{n}{2} + 1$ entries in the second row (and column) are positive. Let

$$M_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

Since M_1 and M_2 are nonsingular, they are not principal submatrices of B . We now show by induction on the number of negative entries in the second row that B contains a row with $\frac{n}{2}$ negative entries. For the base case, first notice that the nonsingularity of B implies that B must have a row with at least one negative entry, as otherwise it will have a repeated row; without loss of generality, we assume that $b_{2n} = -1$. By Lemma 2.10, $B[\{1, \dots, \frac{n}{2} + 1\}]$ has a negative entry; without loss of generality, suppose $b_{34} = -1$. Then, as $B[\{2, 3, 4, n\}] \neq M_2$, either b_{3n} or b_{4n} is negative, implying that either the third or fourth row contains two negative entries. It follows that B must contain a row with two negative entries.

Now, for the inductive step, suppose the second row contains $2 \leq k \leq \frac{n}{2} - 1$ negative entries. Without loss of generality, suppose $b_{2j} = -1$ for $j \in U = \{n - k + 1, \dots, n\}$. As in the base case, Lemma 2.10 implies that $B[\{1, \dots, \frac{n}{2} + 1\}]$ has a negative entry, and, again, without loss of generality, we may assume that $b_{34} = -1$. Since $B[\{1, 2, p, q\}] \neq M_1$ for

$p, q \in U$, $b_{pq} = 1$ for all $p, q \in U$. Similarly, $B[\{1, 3, 4, j\}] \neq M_1$ and $B[\{2, 3, 4, j\}] \neq M_2$ for $j \in U$, implying that $b_{3j} \neq b_{4j}$ for all $j \in U$; so, suppose $b_{3j} = x_j$ and $b_{4j} = -x_j$ for all $j \in U$. Then, as $\ell_6 = \mathbf{N}$, $-16(x_p - x_q)^2 = \det B[\{1, 2, 3, 4, p, q\}] = 0$ for all $p, q \in U$; hence, $x_p = x_q$ for all $p, q \in U$. It follows that either the third or the fourth row contains $(n - (n - k + 1) + 1) + 1 = k + 1$ negative entries. Hence, by induction, B must have a row with $\frac{n}{2}$ negative entries; by Lemma 2.11, B is singular, a contradiction. \square

We note that Theorem 2.12 cannot be generalized for n odd, since, by the Inverse Theorem, ANSNA is attained by $(A(C_n))^{-1}$ (see [3, Observation 3.1]).

Proposition 2.13. *No real symmetric matrix has an epr-sequence starting SSNSNSS...*

Proof. Let $B = [b_{ij}]$ be a real symmetric with an epr-sequence starting SSNSNSS... By the Inheritance Theorem, B has a 7×7 principal submatrix $B[\alpha]$ with epr-sequence $\ell'_1 \ell'_2 \mathbf{N} \ell'_4 \mathbf{N} \ell'_6 \mathbf{A}$. Without loss of generality, suppose $\alpha = \{2, 3, \dots, 8\}$. By the NN Theorem, $\ell'_2, \ell'_4, \ell'_6$ are not \mathbf{N} . Since NAN and NSA are prohibited, $\ell'_4 = \mathbf{S}$ and $\ell'_6 = \mathbf{A}$. Since ANS must be initial, $\ell'_2 = \mathbf{S}$. Hence, $\text{epr}(B[\alpha]) = \ell'_1 \mathbf{S} \mathbf{N} \mathbf{S} \mathbf{N} \mathbf{A} \mathbf{A}$. Since $\text{ASN} \dots \mathbf{A}$ is prohibited, $\ell'_1 \neq \mathbf{A}$. Then, as the epr-sequence SSNSNAA is associated with the pr-sequence 1]1101011, which is unattainable by [1, Proposition 4.1], $\ell'_1 \neq \mathbf{S}$; hence, $\ell'_1 = \mathbf{N}$, so that $\text{epr}(B[\alpha]) = \mathbf{N} \mathbf{S} \mathbf{N} \mathbf{S} \mathbf{N} \mathbf{A} \mathbf{A}$. We note that a simultaneous permutation of the rows and columns of a matrix has no effect on its determinant; thus, since all diagonal entries of $B[\alpha]$ are zero, and because B contains a nonzero diagonal entry, we may assume, without loss of generality, that $b_{11} \neq 0$.

Let $C = B[\{1\} \cup \alpha]$ and $C = [c_{ij}]$. Then $\text{epr}(C)$ starts with \mathbf{S} and $\text{epr}(C[\alpha]) = \text{epr}(B[\alpha]) = \mathbf{N} \mathbf{S} \mathbf{N} \mathbf{S} \mathbf{N} \mathbf{A} \mathbf{A}$. Since every 6×6 principal submatrix of $C[\alpha]$ is nonsingular, $C[\alpha]$ contains at least two nonzero entries in each row (and column), as otherwise $C[\alpha]$ contains a 6×6 principal submatrix with a row (and column) consisting of only zeros. Moreover, we note that $c_{11} = b_{11} \neq 0$; because multiplication of any row and column of a matrix by a nonzero constant preserves the rank of every submatrix, we may assume without loss of generality that $c_{11} = 1$. Since $C[\alpha]$ contains a nonzero principal minor of order 2, we may assume, without loss of generality, that $\det((C[\alpha])[\{1, 2\}]) \neq 0$; thus, $-(c_{23})^2 = C_{23} = (C[\alpha])[\{1, 2\}] \neq 0$; hence, $c_{23} \neq 0$, and, without loss of generality, we may assume that $c_{23} = 1$. Since $C[\alpha]$ contains at least two nonzero entries in each row and column, $c_{2j} \neq 0$ for some $j \in \{4, 5, 6, 7, 8\}$; so, we may assume that $c_{24} = 1$. It follows that $2c_{34} = C_{234} = 0$, and so $c_{34} = 0$. Then, as $C[\alpha]$ contains at least two nonzero entries in each row and column, $c_{3j} \neq 0$ for some $j \in \{5, 6, 7, 8\}$; thus, suppose $c_{35} = 1$. It follows that $2c_{25} = C_{235} = 0$, and so $c_{25} = 0$. Now we have $-1 + 2c_{12}c_{13} = C_{123} = 0$, $-1 + 2c_{12}c_{14} = C_{124} = 0$ and $-1 + 2c_{13}c_{15} = C_{135} = 0$; it follows that c_{12}, c_{13}, c_{14} and c_{15} are nonzero. Let $c_{12} = x$; then $c_{13} = c_{14} = 1/2x$ and $c_{15} = x$. We now show that each of c_{16}, c_{17} and c_{18} is nonzero. Suppose to the contrary that $c_{1j} = 0$ for some $j \in \{6, 7, 8\}$; then $-(c_{ij})^2 = C_{1ij} = 0$ for all $i \in \{3, 4, \dots, 8\} \setminus \{j\}$; hence, $c_{ij} = 0$ for all $i \in \{3, 4, \dots, 8\}$, implying that $C[\alpha]$ contains a row with only one nonzero entry, which is a contradiction. Without loss of generality, we may assume that $c_{16} = c_{17} = c_{18} = 1$. Now, observe that $C_{145} = c_{45}(1 - c_{45})$; since all the principal minors of order 3 are zero, it follows that $c_{45} = 0$ or $c_{45} = 1$. Besides for the $(1, 2)$ -entry x , we have similar restrictions for all the remaining unknown entries of C ; notice that, for $j \in \{6, 7, 8\}$, $C_{12j} = c_{2j}(2x - c_{2j})$, $C_{13j} = c_{3j}(1/x - c_{3j})$, $C_{14j} = c_{4j}(1/x - c_{4j})$ and $C_{15j} = c_{5j}(2x - c_{5j})$. Similarly, for $k \in \{7, 8\}$, $C_{16k} = c_{6k}(2 - c_{6k})$. Lastly, $C_{178} = c_{78}(2 - c_{78})$. It is now clear that, besides the $(1, 2)$ -entry x , each unknown entry of C is restricted to exactly two values.

We now show that $c_{45} = 1$. Suppose to the contrary that $c_{45} = 0$. Since $C[\alpha]$ must contain at least two nonzero entries in each row and column, without loss of generality, we may assume that b_{56} is nonzero, implying that $c_{56} = 2x$. Then $4xc_{36} = C_{356} = 0$, and therefore $c_{36} = 0$. We proceed by examining the only two possibilities for the entry c_{26} . First, suppose $c_{26} = 0$. Since all the principal minors of order 5 of C are zero, $4xc_{46} = C_{23456} = 0$, implying that $c_{46} = 0$. Then $C_{12456} = -4x^2 \neq 0$, a contradiction. So, suppose $c_{26} = 2x$. Since $4xc_{46} = C_{246} = 0$, $c_{46} = 0$. Since $C[\alpha]$ must contain at least two nonzero entries in each row and column, suppose, without loss of generality, that $c_{47} \neq 0$; hence, $c_{47} = 1/x$. Since $2c_{27}/x = C_{247} = 0$, $c_{27} = 0$. Now, observe that $C_{13457} = (-2x + 2x^2c_{37} + c_{57} - xc_{37}c_{57})/2x^3$ and $C_{23457} = 2c_{57}/x - 2c_{37}c_{57}$; since $C_{13457} = 0$, at least one of c_{37} and c_{57} is nonzero; then, as $C_{23457} = 0$, $c_{37} \neq 0$, and so $c_{37} = 1/x$. It follows that $2c_{57}/x = C_{357} = 0$, and so $c_{57} = 0$. As $-4 + 2c_{67} = C_{14567} = 0$, $c_{67} = 2$. Then we have $C_{234567} = 0$, implying that $C[\alpha]$ has a singular 6×6 principal submatrix, which is a contradiction. We conclude that $c_{45} \neq 0$; hence, $c_{45} = 1$.

Now, observe that at least one of c_{36} , c_{37} , c_{38} , c_{46} , c_{47} and c_{48} is nonzero, as otherwise $C[\alpha]$, which is nonsingular, would have two identical rows; thus, without loss of generality, we assume that $c_{36} \neq 0$; hence, $c_{36} = 1/x$. Similarly, at least one of c_{27} , c_{28} , c_{57} and c_{58} is nonzero, as otherwise $C[\{2, 3, 4, 5, 7, 8\}] = (C[\alpha])[\{1, 2, 3, 4, 6, 7\}]$, which is nonsingular, would have two identical rows; without loss of generality, we assume that $c_{27} \neq 0$; thus, $c_{27} = 2x$. Now the conditions $C_{236} = C_{237} = C_{247} = C_{356} = 0$ imply that $c_{26} = c_{37} = c_{47} = c_{56} = 0$.

Finally, we consider the only two possibilities for the entry c_{57} . First, suppose $c_{57} = 2x$. Then $C_{234567} = 0$, a contradiction. Now, suppose $c_{57} = 0$. Since $C_{234567} = -4x^2(c_{46} - 1/x)^2$ is nonzero, $c_{46} = 0$. Then $-2c_{67} = C_{14567} = 0$, and so $c_{67} = 0$. Since every row and column of $C[\alpha]$ must contain at least two nonzero entries, it follows that c_{68} and c_{78} are nonzero, implying that $c_{68} = c_{78} = 2$. The conditions $C_{278} = C_{368} = 0$ imply that $c_{28} = c_{38} = 0$. Hence, $C_{23678} = 16 \neq 0$, a contradiction. \square

3 Pr-sequences not containing three consecutive 1s

We begin with results that forbid certain pr-sequences not containing three consecutive 1s; we then implement these in Theorem 3.10, where, for real symmetric matrices, we classify all the attainable pr-sequences not containing three consecutive 1s.

It is obvious from Theorem 1.1 that, with the exception of the 0th term r'_0 , we can explicitly determine each term in the pr-sequence of the inverse of a nonsingular real symmetric matrix B . The next result demonstrates that, when $n \geq 3$, r'_0 can always be determined from $\text{pr}(B)$ if this sequence does not end with 111.

Remark 3.1. Let $n \geq 3$, B be a nonsingular real symmetric matrix with $\text{pr}(B) = r_0]r_1 \cdots r_{n-1}1$ and r'_0 be the 0th term of $\text{pr}(B^{-1})$.

- (i) If $r_{n-1}r_n = 01$, then $r'_0 = 1$.
- (ii) If $r_{n-2}r_{n-1}r_n = 011$, then $r'_0 = 0$.

(i) is immediate from Theorem 1.1, since B obviously has a principal minor of order $n - 1$ that is zero. As for (ii), first, notice that the penultimate term of $\text{epr}(B)$ must be \mathbf{A} , as \mathbf{NSA} is prohibited; therefore, B does not have a principal minor of order $n - 1$ that is zero, implying that $r'_0 = 0$.

The next proposition generalizes a particular case of [2, Lemma 4.5].

Proposition 3.2. *Let B be a real symmetric matrix with $\text{pr}(B) = r_0]r_1 \cdots r_n$. Suppose that $\text{pr}(B)$ does not contain three consecutive 1s and that $r_0]r_1 \neq 1]1$. Then, for any m with $1 \leq m \leq n$, there exists a principal submatrix B' of B such that $\text{pr}(B') = r_0]r_1 \cdots r_m$.*

Proof. Let $1 \leq m \leq n$. By [2, Lemma 4.5], B has a principal submatrix B' with $\text{pr}(B') = r'_0]r_1 r_2 \cdots r_m$. Since B does not contain both a zero and a nonzero diagonal entry, it follows that $r'_0]r_1 = r_0]r_1$, and therefore $\text{pr}(B') = r_0]r_1 \cdots r_m$. \square

Corollary 3.3. *Let $\sigma = r_0]r_1 \cdots r_n$ be a pr-sequence not containing three consecutive 1s. Suppose $r_0]r_1 \neq 1]1$. If any initial subsequence of σ is unattainable, then σ is unattainable.*

It was shown in [2] that appending 0 to the end of an attainable pr-sequence results in a new attainable pr-sequence; but, what if 0 is appended to an unattainable pr-sequence? For example, if we append 0 to $1]1011$, which is unattainable (see [2, Table 5.4]), we obtain the attainable pr-sequence $1]10110$ (see [2, Table 6.1]). However, there are some cases where appending 0 preserves unattainability. The next observation, a consequence of Corollary 3.3, illustrates this.

Observation 3.4. *Let $r_0]r_1 \cdots r_n$ be an unattainable pr-sequence not containing three consecutive 1s. Suppose $r_0]r_1 \neq 1]1$. Then $r_0]r_1 \cdots r_n 0$ is also unattainable.*

Propositions 3.5 and 3.7 below are corollaries to Theorem 2.12.

Proposition 3.5. *Let B be a real symmetric matrix with $\text{epr}(B) = \mathbf{ANSNSN} \cdots$. Then, for $k \geq 1$, $\ell_{2k} = \mathbf{N}$. Furthermore, $\text{epr}(B) = \mathbf{ANSNSNSN} \overline{\mathbf{N}} \overline{\mathbf{N}}$ or $\text{epr}(B) = \mathbf{ANSNSNSN} \overline{\mathbf{A}}$.*

Proof. Let $k \geq 1$. By hypothesis, the first assertion holds for $k \leq 3$. Suppose $\ell_{2k} \neq \mathbf{N}$ for some $k > 3$. By the Inheritance Theorem, B has a nonsingular $2k \times 2k$ principal submatrix with epr-sequence $\mathbf{ANXNYN} \cdots \mathbf{A}$, where $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \{\mathbf{A}, \mathbf{S}, \mathbf{N}\}$. By the \mathbf{NN} Theorem, \mathbf{X} and \mathbf{Y} are not \mathbf{N} . Since \mathbf{NAN} is prohibited, $\mathbf{X} = \mathbf{Y} = \mathbf{S}$, a contradiction to Theorem 2.12. The final assertion is immediate from the \mathbf{NN} Theorem and the fact that \mathbf{NAN} is prohibited. \square

Corollary 3.6. *The pr-sequence $0]1010101\overline{0}1\overline{0}$ is not attainable by a real symmetric matrix.*

Proof. Since $0]1010101\overline{0}1$ satisfies the hypothesis of Observation 3.4, it suffices to show that this sequence is not attainable. Suppose that there is a real symmetric matrix B with $\text{pr}(B) = 0]1010101\overline{0}1$ and $\text{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$. Obviously, $\ell_1 = \ell_n = \mathbf{A}$ and $\ell_2 = \ell_4 = \ell_6 = \mathbf{N}$. Since \mathbf{NAN} is prohibited, $\ell_3 = \ell_5 = \mathbf{S}$. Hence, $\text{epr}(B) = \mathbf{ANSNSN} \cdots \mathbf{XA}$, where \mathbf{X} is not \mathbf{N} , which contradicts Proposition 3.5. \square

Proposition 3.7. *Let B be a real symmetric matrix with $\text{epr}(B) = \text{SNSNSN} \cdots$. Then, for $k \geq 1$, $\ell_{2k} = \text{N}$. Furthermore, $\text{epr}(B) = \text{SNSNSN}\overline{\text{SN}}\overline{\text{N}}$ or $\text{epr}(B) = \text{SNSNSN}\overline{\text{SNA}}$.*

Proof. Let $k \geq 1$. By hypothesis, the first assertion holds for $k \leq 3$. Suppose $\ell_{2k} \neq \text{N}$ for some $k > 3$. By the Inheritance Theorem, B has a nonsingular $2k \times 2k$ principal submatrix with an epr-sequence $\text{XNYNZN} \cdots \text{A}$, where $\text{X}, \text{Y}, \text{Z} \in \{\text{A}, \text{S}, \text{N}\}$. By the NN Theorem, X, Y and Z are not N . Since NAN is prohibited, $\text{Y} = \text{Z} = \text{S}$. Since $\text{SN} \cdots \text{A} \cdots$ is prohibited, $\text{X} \neq \text{S}$, and hence $\text{X} = \text{A}$, a contradiction to Theorem 2.12. As in Proposition 3.5, the final assertion follows from the NN Theorem and the fact that NAN is prohibited. \square

Corollary 3.8. *The pr-sequence $1]1010101\overline{0}110\overline{0}$ is not attainable by a real symmetric matrix.*

Proof. Suppose there is a real symmetric matrix B with $\text{pr}(B) = 1]1010101\overline{0}110\overline{0}$. Let $\text{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$. Obviously, $\ell_1 = \text{S}$ and $\ell_2 = \ell_4 = \ell_6 = \text{N}$. Since NAN is prohibited, $\ell_3 = \ell_5 = \text{S}$. Hence, $\text{epr}(B) = \text{SNSNSN} \cdots \text{XYNN}$, where X and Y are both not N , which contradicts Proposition 3.7. \square

Before proving the main result of this section, we need a lemma.

Lemma 3.9. *Let B be a real symmetric matrix with $\text{pr}(B) = r_0]r_1 \cdots r_n$. Suppose $r_1 r_2 \cdots r_n$ does not contain three consecutive 1s. Let $1 \leq k \leq \text{rank}(B) - 2$. If $r_k r_{k+1} = 01$, then either $r_{k+2} r_{k+3} \cdots r_n = \overline{0}11\overline{0}$ or $r_{k+2} r_{k+3} \cdots r_n = \overline{0}101\overline{0}$*

Proof. Suppose $r_k r_{k+1} = 01$. We proceed by examining the only two possibilities for r_{k+2} .

Case 1: $r_{k+2} = 1$. Now we have $r_k r_{k+1} r_{k+2} = 011$. If $n = k + 2$, then we are done. Now, suppose $n > k + 2$. By hypothesis, $r_{k+3} = 0$, and therefore, by the 0110 Theorem, $r_{k+2} r_{k+3} \cdots r_n = 1\overline{0}$, where $\overline{0}$ is non-empty.

Case 2: $r_{k+2} = 0$. Now we have $r_k r_{k+1} r_{k+2} = 010$. Then, as $\text{rank}(B) \geq k + 2$, by the 00 Theorem, $r_{k+3} \neq 0$; hence, $r_{k+3} = 1$, and so $r_{k+2} r_{k+3} = 01$. If $n = k + 3$, then we are done. Suppose $n > k + 3$. If $\text{rank}(B) = k + 3$, then we have $r_{k+2} r_{k+3} \cdots r_n = 01\overline{0}$, where $\overline{0}$ is non-empty. Suppose $\text{rank}(B) > k + 3$, i.e., suppose $\text{rank}(B) \geq k + 4$. Thus, so far we have $r_k r_{k+1} r_{k+2} r_{k+3} = 0101$, where $r_{k+2} r_{k+3} = 01$ and $1 \leq k + 2 \leq \text{rank}(B) - 2$. Since n is finite, it is evident that reimplementing the steps above by replacing k with $k + 2$, and repeating this process until reaching the last term of the sequence, yields the desired conclusion. \square

With the next theorem, we classify all the attainable pr-sequences of order $n \geq 3$ not containing three consecutive 1s.

Theorem 3.10. *Let $n \geq 3$. A pr-sequence of order n not containing three consecutive 1s is attainable by a real symmetric matrix if and only if it is one of the following sequences.*

1. $0]100\overline{0}$.
2. $0]1\overline{0}101\overline{0}$.
3. $0]1011\overline{0}$.
4. $0]101011\overline{0}$.

5. $0]110\bar{0}$.
6. $0]1101\bar{0}$.
7. $0]11011\bar{0}$.
8. $1]000\bar{0}$.
9. $1]010\bar{0}$.
10. $1]01\bar{0}101\bar{0}$.
11. $1]01\bar{0}11\bar{0}$.
12. $1]100\bar{0}$.
13. $1]1\bar{0}1010\bar{0}$.
14. $1]10110\bar{0}$.
15. $1]1010110\bar{0}$.

Proof. Let B be a real symmetric matrix with $\text{pr}(B) = r_0]r_1 \cdots r_n$ not containing three consecutive 1s. Since $0]0 \cdots$ is forbidden by definition, $r_0]r_1 \in \{0]1, 1]0, 1]1\}$. We proceed by examining all the possibilities for $r_0]r_1r_2$.

Case i: $r_0]r_1r_2 = 0]10$. If $r_3 = 0$, then, by the 00 Theorem, we have sequence (1). Suppose $r_3 = 1$. Hence, $\text{pr}(B)$ starts $0]101 \cdots$. If $\text{rank}(B) = 3$, then $\text{pr}(B) = 0]101\bar{0}$, which is sequence (2). Now, suppose $\text{rank}(B) > 3$. Then $r_2r_3 = 01$ and $1 \leq 2 \leq \text{rank}(B) - 2$; hence, by applying Lemma 3.9 to $\text{pr}(B)$, starting with $k = 2$, we have either $\text{pr}(B) = 0]101\bar{0}10\bar{1}\bar{0}$ or $\text{pr}(B) = 0]101\bar{0}11\bar{0}$. Hence, by Corollary 3.6, $\text{pr}(B)$ is one of the sequences (2), (3) and (4).

Case ii: $r_0]r_1r_2 = 0]11$. By hypothesis, $r_3 = 0$. If $\text{rank}(B) = 2$, then $\text{pr}(B) = 0]110\bar{0}$, which is sequence (5). Now suppose $\text{rank}(B) > 2$. Then $n > 3$ and, by the 00 Theorem, $r_4 \neq 0$, implying that $r_4 = 1$. Hence, $\text{pr}(B)$ starts $0]1101 \cdots$. If $n = 4$, then we have sequence (6). Suppose $n > 4$. If $r_5 = 1$, then, by the 0110 Theorem, we must have sequence (7), where $\bar{0}$ may be empty. Now, suppose $r_5 = 0$. If $n = 5$, then we have sequence (6). Suppose $n > 5$. Thus far we have $\text{pr}(B) = 0]11010 \cdots$; it follows from [2, Theorem 7.2] that $r_6 = 0$, and therefore, by the 00 Theorem, we have sequence (6).

Case iii: $r_0]r_1 = 1]0$. If $r_2 = 0$, then, by the 00 Theorem, we have sequence (8). Now, suppose $r_2 = 1$. Hence, $\text{pr}(B)$ starts $1]01 \cdots$. If $\text{rank}(B) = 2$, then $\text{pr}(B) = 1]010\bar{0}$, which is sequence (9). Now, suppose $\text{rank}(B) > 2$. Then $r_1r_2 = 01$ and $1 \leq 1 \leq \text{rank}(B) - 2$; hence, by applying Lemma 3.9 to $\text{pr}(B)$, starting with $k = 1$, we have either $\text{pr}(B) = 1]01\bar{0}101\bar{0}$ or $\text{pr}(B) = 1]01\bar{0}11\bar{0}$. Thus, $\text{pr}(B)$ is either sequence (10) or (11).

Case iv: $r_0]r_1 = 1]1$. By hypothesis, $r_2 = 0$. If $r_3 = 0$, then the 00 Theorem implies that we have sequence (12). Now, suppose $r_3 = 1$. Hence, $\text{pr}(B)$ starts $1]101 \cdots$. Suppose $\text{rank}(B) = 3$; then $\text{pr}(B) = 1]101\bar{0}$, and, by [2, Theorem 4.1], $\bar{0}$ is non-empty, implying that $\text{pr}(B) = 1]1010\bar{0}$, which is sequence (13). Now, suppose $\text{rank}(B) > 3$. Then $r_2r_3 = 01$ and $1 \leq 2 \leq \text{rank}(B) - 2$; hence, by applying Lemma 3.9 to $\text{pr}(B)$, starting with $k = 2$, we have

$\text{pr}(B) = 1]101\overline{0}101\overline{0}$ or $\text{pr}(B) = 1]101\overline{0}11\overline{0}$; again, it follows from [2, Theorem 4.1] that in either case $\overline{0}$ must be non-empty, and therefore $\text{pr}(B) = 1]101\overline{0}1010\overline{0}$ or $\text{pr}(B) = 1]101\overline{0}110\overline{0}$. Hence, by Corollary 3.8, $\text{pr}(B)$ is one of the sequences (13), (14) and (15).

For the other direction, since appending 0 to the end of an attainable sequence results in another attainable sequence (see [2, Theorem 2.6]), it suffices to establish the attainability of each sequence when $\overline{0}$ is empty. We assume that the sequence under consideration has order $n \geq 3$ and provide an $n \times n$ real symmetric matrix that attains it.

1. $0]100\overline{0}$: $\text{pr}(J_3) = 0]100$.
2. $0]1\overline{0}101\overline{0}$: $\text{pr}((A(C_n))^{-1}) = 0]1\overline{0}101$, with n odd (see [2, Lemma 3.4] and Remark 3.1).
3. $0]1011\overline{0}$: $\text{pr}(J_4 - 2I_4) = 0]1011$.
4. $0]101011\overline{0}$: $\text{pr}(M_{0101011}) = 0]101011$, where $M_{0101011}$ appears in [2, p. 2153].
5. $0]110\overline{0}$: $\text{pr}(J_1 \oplus J_2) = 0]110$.
6. $0]1101\overline{0}$: $\text{pr}(J_4 - 3I_4) = 0]1101$.
7. $0]11011\overline{0}$: $\text{pr}(J_5 - 3I_5) = 0]11011$.
8. $1]000\overline{0}$: $\text{pr}(0_3) = 1]000$.
9. $1]010\overline{0}$: $\text{pr}((J_2 - I_2) \oplus 0_1) = 1]010$.
10. $1]01\overline{0}101\overline{0}$: $\text{pr}(A(P_n)) = 1]01\overline{0}101$, with n even (see [2, Lemma 3.3]).
11. $1]01\overline{0}11\overline{0}$: $\text{pr}(A(C_n)) = 1]01\overline{0}11$, with n odd (see [2, Lemma 3.4]).
12. $1]100\overline{0}$: $\text{pr}(J_1 \oplus 0_2) = 1]100$.
13. $1]1\overline{0}1010\overline{0}$: $\text{pr}((A(C_{n-1}))^{-1} \oplus 0_1) = 1]1\overline{0}1010$, with n even (see [2, Lemma 3.4], Remark 3.1 and [2, Theorem 2.3]).
14. $1]10110\overline{0}$: $\text{pr}((J_4 - 2I_4) \oplus 0_1) = 1]10110$.
15. $1]1010110\overline{0}$: $\text{pr}(M_{0101011} \oplus 0_1) = 1]1010110$, where $M_{0101011}$ appears in [2, p. 2153].

That concludes the proof. \square

We conclude this section with a classification of the attainable pr -sequences that only contain three consecutive 1s in the initial subsequence $1]11$. The primary motivation for including this result is its application in Section 4.

Proposition 3.11. *The epr-sequences $\text{SSNSNS}\overline{\text{NS}}\text{SN}\overline{\text{N}}$ and $\text{SSNS}\overline{\text{NS}}\text{NAA}$ are not attainable by a real symmetric matrix.*

Proof. Suppose to the contrary that there is a real symmetric matrix B with $\text{epr}(B) = \text{SSNSNS}\overline{\text{SSNN}}$. Notice that $\text{rank}(B)$ is odd. If $\overline{\text{NS}}$ is empty, then we have a contradiction to Proposition 2.13. So, suppose $\overline{\text{NS}}$ is non-empty. Let $B[\alpha]$ be a nonsingular 1×1 principal submatrix of B . By the Schur Complement Theorem, $\text{rank}(B/B[\alpha])$ is even, $\text{rank}(B/B[\alpha]) \geq 8$, and $\text{epr}(B/B[\alpha]) = \text{XNYNZN}\cdots$, where $X, Y, Z \in \{A, S, N\}$. Then, as $\text{rank}(B/B[\alpha]) \geq 8$, by the NN Theorem, X, Y and Z are not N . Since NAN is prohibited, $Y = Z = S$. Thus, we have $\text{epr}(B/B[\alpha]) = \text{XNSNSN}\cdots$, where X is not N . It follows from Propositions 3.5 and 3.7 that $\text{rank}(B/B[\alpha])$ is odd, a contradiction.

Now, suppose $\text{SSNSNS}\overline{\text{NAA}}$ is attainable. Then applying [3, Observation 2.19(2)] to this sequence implies that $\text{SSNSNS}\overline{\text{SSN}}$ is attainable, a contradiction to the first assertion. \square

Corollary 3.12. *The pr-sequence $1]110101\overline{0110}$ is not attainable by a real symmetric matrix.*

Proof. Suppose that there is a real symmetric matrix B with $\text{pr}(B) = 1]110101\overline{0110}$ and $\text{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Obviously, $\ell_1 = S$ and $\ell_3 = \ell_5 = N$. By the NN Theorem, and because NAN is prohibited, $\ell_4 = S$. Since ℓ_2 is not N , it follows from Proposition 2.3 that $\ell_2 = S$. Hence, $\text{epr}(B) = \text{SSNSN}\cdots$. We examine two cases.

Case 1: $\overline{0}$ is empty. Notice that $\text{pr}(B) = 1]110101\overline{011} = 1]1101\overline{01011}$. Moreover, $\ell_n = A$ and $\ell_i = N$ for all odd i with $3 \leq i \leq n-2$. Then, as NAN is prohibited, $\ell_j = S$ for all even j with $4 \leq j \leq n-3$. Therefore, we have $\text{epr}(B) = \text{SSNSNS}\overline{\text{NXA}}$, where X is not N . Since NSA is prohibited, $X = A$, which contradicts Proposition 3.11.

Case 2: $\overline{0}$ is non-empty. Thus, $\text{pr}(B) = 1]110101\overline{01100} = 1]1101\overline{0101100}$. As in the preceding case, the fact that NAN is prohibited implies that $\text{epr}(B) = \text{SSNSNS}\overline{\text{NXYN}}$, where X and Y are not N . By Theorem 2.7, $X = S$. Then, as NSA is prohibited, $Y = S$. Hence, $\text{epr}(B) = \text{SSNSNS}\overline{\text{SSNN}}$, a contradiction to Proposition 3.11. \square

Proposition 3.13. *Let $n \geq 3$. A pr-sequence $r_0]r_1\cdots r_n$, with $r_1r_2\cdots r_n$ not containing three consecutive 1s, is attainable by a real symmetric matrix if and only if it is one of the sequences in Theorem 3.10 or one of the following sequences.*

16. $1]110\overline{0}$.

17. $1]110\overline{01010}$.

18. $1]11011\overline{0}$.

Proof. Let B be a real symmetric matrix with $\text{pr}(B) = r_0]r_1\cdots r_n$. Suppose $r_1r_2\cdots r_n$ does not contain three consecutive 1s. If $r_0]r_1r_2 \neq 1]11$, then $\text{pr}(B)$ does not contain three consecutive 1s, and therefore it is one of the sequences listed in Theorem 3.10. Thus, suppose $r_0]r_1r_2 = 1]11$. By hypothesis, $r_3 = 0$. If $n = 3$, then $\text{pr}(B)$ is sequence (16). So, suppose $n > 3$. If $r_4 = 0$, then, by the 00 Theorem, $\text{pr}(B)$ is sequence (16). Now, suppose $r_4 = 1$. Then $\text{pr}(B)$ starts $1]1101\cdots$. If $\text{rank}(B) = 4$, then $\text{pr}(B) = 1]1101\overline{0}$, which is sequence (17). Now, suppose $\text{rank}(B) > 4$. Hence, $r_3r_4 = 01$ and $1 \leq 3 \leq \text{rank}(B) - 2$. It follows from applying Lemma 3.9 to $\text{pr}(B)$, starting with $k = 3$, that $\text{pr}(B) = 1]1101\overline{01010}$ or $\text{pr}(B) = 1]1101\overline{0110}$. Hence, by Corollary 3.12, $\text{pr}(B)$ is either sequence (17) or sequence (18).

For the other direction, as in Theorem 3.10, it suffices to show that each sequence is attainable when $\bar{0}$ is empty. By [2, Theorem 3.7], the sequences 1]110 and 1]11011 are attainable by $Q_{3,1}$ and $Q_{5,1}$, respectively. Finally, 1]110 $\bar{1}$ 01 is attained by $(A(F_n))^{-1}$ (see [2, Lemma 3.5]), where n is even and F_n is the graph on n vertices formed by adding a pendent edge to C_{n-1} . \square

4 Epr-sequences with an N in every subsequence of length 3

This section focuses on epr-sequences with an N in every subsequence of length 3, and culminates with a classification of all the attainable epr-sequences with this property.

The sequence accounted for in the next result is of particular relevance to the main result at the end of this section.

Proposition 4.1. *Let $n \geq 3$ and $B = [b_{ij}]$ be the $n \times n$ real symmetric matrix with $b_{ij} = (i - j)^2$. Then $\text{epr}(B) = \mathbf{NAA}\bar{\mathbf{N}}$.*

Proof. Suppose that $\text{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$. It is easy to verify the assertion for $n = 3$. Suppose $n > 3$. Obviously, $\ell_1 = \mathbf{N}$. Let $p, q, r, s \in \{1, 2, \dots, n\}$, where $p < q < r < s$. Since every off-diagonal entry of B is nonzero, we have $B_{pq} = -(b_{pq})^2 \neq 0$ and $B_{pqr} = 2b_{pq}b_{pr}b_{qr} \neq 0$. A simple computation reveals that the order-4 principal minor B_{pqrs} is given by

$$\begin{aligned} & (b_{ps}b_{qr})^2 + (b_{pr}b_{qs})^2 + (b_{pq}b_{rs})^2 - 2b_{pr}b_{ps}b_{qr}b_{qs} - 2b_{pq}b_{ps}b_{qr}b_{rs} - 2b_{pq}b_{pr}b_{qs}b_{rs} = \\ & ((p-s)(q-r))^4 + ((p-r)(q-s))^4 + ((p-q)(r-s))^4 \\ & - 2((p-r)(p-s)(q-r)(q-s))^2 - 2((p-q)(p-s)(q-r)(r-s))^2 \\ & - 2((p-q)(p-r)(q-s)(r-s))^2 = 0. \end{aligned}$$

Hence, we have $\ell_2 = \ell_3 = \mathbf{A}$ and $\ell_4 = \mathbf{N}$. The conclusion now follows from Proposition 2.4. \square

Observation 4.2. *If an attainable pr-sequence does not contain three consecutive 1s, then an attainable epr-sequence associated with it contains an N in every subsequence of length 3.*

Remark 4.3. The converse of Observation 4.2 is false. An attainable epr-sequence starting $\mathbf{SS} \cdots$, or starting $\mathbf{SA} \cdots$, with an N in every subsequence of length 3, provides a counterexample. It can be deduced that all counterexamples are of that form, and therefore that the converse of Observation 4.2 is true if additionally we assume that the pr-sequence does not start with 1]11.

Observation 4.4. *Let $n \geq 3$ and B be a real symmetric matrix with $\text{pr}(B) = r_0]r_1 \cdots r_n$. Then $\text{epr}(B)$ contains an N in every subsequence of length 3 if and only if $r_1 r_2 \cdots r_n$ does not contain three consecutive 1s*

Observation 4.4 suggests that we can use Theorem 3.10 and Proposition 3.13 to classify all the epr-sequences with an N in every subsequence of length 3, as the pr-sequences associated with these epr-sequences must be those listed on these results.

Theorem 4.5. *Let $n \geq 3$. An epr-sequence of order n with an \mathbf{N} in every subsequence of length 3 is attainable by a real symmetric matrix if and only if it is one of the following sequences.*

1. $\mathbf{ANN}\bar{\mathbf{N}}$.
- 2a. $\mathbf{AN}\bar{\mathbf{S}}\mathbf{NA}$.
- 2b. $\mathbf{AN}\bar{\mathbf{S}}\mathbf{NSN}\bar{\mathbf{N}}$.
- 3a. \mathbf{ANAA} .
- 3b. $\mathbf{ANSSN}\bar{\mathbf{N}}$.
- 4a. \mathbf{ANSNAA} .
- 4b. $\mathbf{ANSNSSN}\bar{\mathbf{N}}$.
- 5a. $\mathbf{AAN}\bar{\mathbf{N}}$.
- 5b. $\mathbf{ASN}\bar{\mathbf{N}}$.
- 6a. \mathbf{AANA} .
- 6b. $\mathbf{ASNSN}\bar{\mathbf{N}}$.
- 7a. \mathbf{AANAA} .
- 7b. $\mathbf{ASNSSN}\bar{\mathbf{N}}$.
8. $\mathbf{NNN}\bar{\mathbf{N}}$.
9. $\mathbf{NSN}\bar{\mathbf{N}}$.
- 10a. $\mathbf{NS}\bar{\mathbf{S}}\mathbf{NA}$.
- 10b. $\mathbf{NS}\bar{\mathbf{S}}\mathbf{NSN}\bar{\mathbf{N}}$.
- 11a. $\mathbf{N}\bar{\mathbf{S}}\mathbf{NAA}$.
- 11b. $\mathbf{N}\bar{\mathbf{S}}\mathbf{NSSN}\bar{\mathbf{N}}$.
- 11c. $\mathbf{NAAN}\bar{\mathbf{N}}$.
12. $\mathbf{SNN}\bar{\mathbf{N}}$.
13. $\mathbf{S}\bar{\mathbf{N}}\bar{\mathbf{S}}\mathbf{NSN}\bar{\mathbf{N}}$.
14. $\mathbf{SNSSN}\bar{\mathbf{N}}$.
15. $\mathbf{SNSNSSN}\bar{\mathbf{N}}$.
- 16a. $\mathbf{SAN}\bar{\mathbf{N}}$.

16b. $SSN\bar{N}$.

17a. $SS\bar{N}SNA$.

17b. $SS\bar{N}S\bar{N}S\bar{N}$.

18a. $SSNAA$.

18b. $SSN\bar{S}S\bar{N}$.

Proof. Let B be a real symmetric matrix with $\text{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose that $\text{epr}(B)$ contains an N in every subsequence of length 3. It follows from Observation 4.4 that $\text{pr}(B)$ is one of the sequences listed in Theorem 3.10 or Proposition 3.13. We examine the 18 possible cases.

Case 1: $\text{pr}(B) = 0]100\bar{0}$. Obviously, $\text{epr}(B) = ANN\bar{N}$, which is sequence (1).

Case 2: $\text{pr}(B) = 0]1\bar{0}101\bar{0}$. First, suppose $\bar{0}$ is empty. Then, as NAN is prohibited, $\text{epr}(B) = A\bar{N}SNA$, which is sequence (2a). Now, suppose $\bar{0}$ is non-empty. Similarly, since NAN is prohibited, $\text{epr}(B) = A\bar{N}S\bar{N}S\bar{N}$, which is sequence (2b).

Case 3: $\text{pr}(B) = 0]1011\bar{0}$. If $\bar{0}$ is empty, then, as NSA is prohibited, $\text{epr}(B) = ANAA$, which is sequence (3a). If $\bar{0}$ is non-empty, then, since NSA and NAS are prohibited, we must have $ANSSN\bar{N}$ or $ANAAN\bar{N}$; as the latter sequence is forbidden by Theorem 2.6, $\text{epr}(B)$ is sequence (3b).

Case 4: $\text{pr}(B) = 0]101011\bar{0}$. Suppose $\bar{0}$ is empty. Since NAN and NSA are prohibited, $\text{epr}(B) = ANSNA\bar{A}$, which is sequence (4a). Now suppose $\bar{0}$ is non-empty. Then, as NAN , NAS and NSA are prohibited, $\text{epr}(B)$ is either $ANSN\bar{S}S\bar{N}$ or $ANSNAAN\bar{N}$; by Theorem 2.7, the latter sequence is forbidden, and thus we have sequence (4b).

Case 5: $\text{pr}(B) = 0]110\bar{0}$. Clearly, $\text{epr}(B) = AAN\bar{N}$ or $\text{epr}(B) = ASN\bar{N}$, which are sequences (5a) and (5b), respectively.

Case 6: $\text{pr}(B) = 0]1101\bar{0}$. If $\bar{0}$ is empty, then, as $ASNA$ is forbidden, $\text{epr}(B) = AANA$, which is sequence (6a). Suppose $\bar{0}$ is non-empty. Since NAN is prohibited, and because ANS must be initial, $\text{epr}(B) = ASNSN\bar{N}$, which is sequence (6b).

Case 7: $\text{pr}(B) = 0]11011\bar{0}$. Suppose $\bar{0}$ is empty. Since NSA and $ASN\cdots A$ are prohibited, $\text{epr}(B) = AANAA$, which is sequence (7a). Suppose $\bar{0}$ is non-empty. Moreover, suppose $\ell_2 = A$. Obviously, $\ell_n = N$; but, as ANS must be initial, $\ell_4 = A$, and therefore Theorem 2.6 implies that $\ell_n = A$, a contradiction. It follows that we must have $\ell_2 = S$. Since $ASN\cdots A\cdots$ is prohibited, $\text{epr}(B) = ASN\bar{S}S\bar{N}$, which is sequence (7b).

Case 8: $\text{pr}(B) = 1]000\bar{0}$. Clearly, $\text{epr}(B) = NNN\bar{N}$, which is sequence (8).

Case 9: $\text{pr}(B) = 1]010\bar{0}$. Since NAN is prohibited, $\text{epr}(B) = NSN\bar{N}$, which is sequence (9).

Case 10: $\text{pr}(B) = 1]01\bar{0}101\bar{0}$. If $\bar{0}$ is empty, then, as NAN is prohibited, $\text{epr}(B) = NS\bar{N}SNA$, which is sequence (10a). Similarly, if $\bar{0}$ is non-empty, $\text{epr}(B) = NS\bar{N}S\bar{N}S\bar{N}$, which is sequence (10b).

Case 11: $\text{pr}(B) = 1]01\bar{0}11\bar{0}$. First, observe that $\text{pr}(B) = 1]0\bar{1}011\bar{0}$. Suppose $\bar{0}$ is empty. Since NSA and NAN are prohibited, $\text{epr}(B) = N\bar{S}NAA$, which is sequence (11a). Suppose $\bar{0}$ is non-empty. Moreover, suppose $\bar{1}\bar{0}$ is empty. Then, as NAS and NSA are prohibited, $\text{epr}(B)$ is $NSSN\bar{N}$ or $NAAN\bar{N}$, which are sequences (11b) and (11c), respectively. Finally, suppose $\bar{1}\bar{0}$ is non-empty. Since NAS , NSA and NAN are prohibited, $\text{epr}(B)$ is either $NSN\bar{S}N\bar{S}S\bar{N}$ or $NSN\bar{S}NAAN\bar{N}$;

by Theorem 2.7, the latter sequence is forbidden, and therefore $\text{epr}(B)$ is sequence (11b), with \overline{SN} non-empty.

Case 12: $\text{pr}(B) = 1]100\overline{0}$. Obviously, $\text{epr}(B) = \text{SNN}\overline{N}$, which is sequence (12).

Case 13: $\text{pr}(B) = 1]10\overline{1}010\overline{0}$. Since $\text{SN} \cdots \text{A} \cdots$ is prohibited, it is immediate that $\text{epr}(B) = \text{S}\overline{\text{N}}\overline{\text{S}}\text{N}\text{S}\text{N}\overline{\text{N}}$, which is sequence (13).

Case 14: $\text{pr}(B) = 1]10110\overline{0}$. As in Case 13, since $\text{SN} \cdots \text{A} \cdots$ is prohibited, we must have $\text{epr}(B) = \text{SNS}\text{S}\text{N}\overline{\text{N}}$, which is sequence (14).

Case 15: $\text{pr}(B) = 1]1010110\overline{0}$. Again, as $\text{SN} \cdots \text{A} \cdots$ is prohibited, we must have $\text{epr}(B) = \text{SNSN}\text{S}\text{S}\text{N}\overline{\text{N}}$, which is sequence (15).

Case 16: $\text{pr}(B) = 1]110\overline{0}$. Clearly, $\text{epr}(B)$ is either $\text{SAN}\overline{\text{N}}$ or $\text{SSN}\overline{\text{N}}$, which are sequences (16a) and (16b), respectively.

Case 17: $\text{pr}(B) = 1]110\overline{1}010\overline{0}$. Since $\text{SAN} \cdots \text{A} \cdots$ and $\text{SAN} \cdots \text{S} \cdots$ are prohibited by Proposition 2.3, $\ell_2 = \text{S}$. Suppose $\overline{0}$ is empty. Then, as NAN is prohibited, $\text{epr}(B) = \text{SS}\overline{\text{N}}\overline{\text{S}}\text{NA}$, which is sequence (17a). Suppose $\overline{0}$ is non-empty. Similarly, since NAN is prohibited, $\text{epr}(B) = \text{SS}\overline{\text{N}}\overline{\text{S}}\text{N}\text{S}\text{N}\overline{\text{N}}$, which is sequence (17b).

Case 18: $\text{pr}(B) = 1]110110\overline{0}$. As in the preceding case, we must have $\ell_2 = \text{S}$. Suppose $\overline{0}$ is empty. Since NSA is prohibited, $\text{epr}(B) = \text{SSNA}\overline{\text{A}}$, which is sequence (18a). Suppose $\overline{0}$ is non-empty. Hence, the fact that NAS and NSA are prohibited implies that $\text{epr}(B)$ is either $\text{SSN}\text{S}\text{S}\text{N}\overline{\text{N}}$ or $\text{SSNA}\text{AN}\overline{\text{N}}$; by Theorem 2.7, the latter sequence is forbidden, and thus $\text{epr}(B)$ is sequence (18b).

For the other direction, we show that all the sequences listed are attainable, and assume that the sequence under consideration has order $n \geq 3$. Sequence (1) is attained by J_n . Sequence (2a) is attained by $A((C_n)^{-1})$ (see [3, Observation 3.1] and the Inverse Theorem), when $\overline{\text{NS}}$ is non-empty, and by [3, Proposition 2.17], when $\overline{\text{NS}}$ is empty. As for (2b), applying [3, Observation 2.19(1)] to (2a), results in this sequence. Sequence (3a) is attainable by [3, Proposition 2.17]. Sequence (3b) is attainable by applying [3, Observation 2.19(1)] to (3a). Sequence (4a) is attainable by [3, Table 1], and (4b) results from applying [3, Observation 2.19(1)] to (4a). Sequences (5a) and (5b) are attainable by [3, Theorem 4.6]. Sequence (6a) is attainable by [3, Proposition 2.17], and (6b) results from applying [3, Observation 2.19(1)] to (6a). Sequence (7a) is attainable by [3, Proposition 2.17], and (7b) results from applying [3, Observation 2.19(1)] to (7a). Sequence (8) is attained by 0_n . As for (9), applying [3, Observation 2.19(1)] to the sequence NA , which is attained by $J_2 - I_2$, results in this sequence. Sequence (10a) is attainable by [3, Observation 3.1], and (10b) results from applying [3, Observation 2.19(1)] to (10a). Sequence (11a) is attainable by [3, Observation 3.1], while (11b) is obtained from applying [3, Observation 2.19(1)] to (11a). Sequence (11c) is attainable by Proposition 4.1. Sequence (12) is attainable by [3, Theorem 4.6]. Sequences (13), (14) and (15) result from applying [3, Observation 2.19(2)] to (2a), (3a) and (4a), respectively. Sequences (16a) and (16b) are attainable by [3, Theorem 4.6]. According to Proposition 3.13, the sequence $1]110\overline{1}01$ is attainable; by Proposition 2.3, and because NAN is prohibited, an attainable epr -sequence associated with this pr -sequence, must be $\text{SS}\overline{\text{N}}\overline{\text{S}}\text{NA}$, which is sequence (17a). Sequence (17b) results from applying [3, Observation 2.19(2)] to (17a). Sequence (18a) is attainable by [3, Table 5], and (18b) is attainable by [3, Corollary 2.20(2)]. \square

If an epr -sequence is attainable, then the pr -sequence associated with it must be attain-

able. The converse is not true; this is because an epr-sequence associated with a pr-sequence may not be unique, since a 1 in the pr-sequence can correspond to an A or S in the epr-sequence. For example, the epr-sequences NSSN and NAAN, which are associated with the pr-sequence 1]0110, are each attainable by a real symmetric matrix (see [3, Table 4]). We now show that, for real symmetric matrices, almost all attainable pr-sequences not containing three consecutive 1s are associated with a unique epr-sequence.

Proposition 4.6. *Let $n \geq 3$ and σ be a pr-sequence that is attainable by an $n \times n$ real symmetric matrix. Suppose σ does not contain three consecutive 1s, $\sigma \neq 0]110\bar{0}$ and that $\sigma \neq 1]0110\bar{0}$. Then there is a unique attainable epr-sequence associated with σ .*

Proof. Since the attainable epr-sequences associated with pr-sequences not containing three consecutive 1s are the epr-sequences (1a)–(15) listed in Theorem 4.5, an attainable epr-sequence associated with σ must be one of these sequences. Note that σ is not associated with any of the epr-sequences (16a)–(18b), as these are the epr-sequences that are associated with the pr-sequences listed in Proposition 3.13. We consider two cases.

Case 1: $\sigma = 1]010\bar{1}0110\bar{0}$. Observe that σ is associated with the epr-sequence (11b) in Theorem 4.5, with \overline{SN} non-empty. It is easy to see that σ is not associated with any of the other epr-sequences listed in Theorem 4.5, thereby establishing the uniqueness of the associated epr-sequence (11b).

Case 2: $\sigma \neq 1]010\bar{1}0110\bar{0}$. Then, as $\sigma \neq 1]0110\bar{0}$, the epr-sequences (11b) and (11c) in Theorem 4.5 are not associated with σ . Also, it is clear that σ is not associated with the epr-sequence (11a) in Theorem 4.5. Since $\sigma \neq 0]110\bar{0}$, the epr-sequences (5a) and (5b) in Theorem 4.5 are not associated with σ . Thus far we have that σ is not associated with any of the epr-sequences (5a), (5b), (11a), (11b) or (11c). Hence, σ must be one of the pr-sequences (1)–(4), (6)–(10) or (12)–(15) in Theorem 3.10. Now, by considering all the possible cases, one easily verifies that an attainable epr-sequence associated with σ , which must be listed in Theorem 4.5, is unique. \square

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